

# Braided quantum field theories and their symmetries

Yuya Sasai <sup>\*</sup> and Naoki Sasakura <sup>†</sup>

*Yukawa Institute for Theoretical Physics, Kyoto University,  
Kyoto 606-8502, Japan*

## Abstract

Braided quantum field theories proposed by Oeckl can provide a framework for defining quantum field theories having Hopf algebra symmetries. In quantum field theories, symmetries lead to non-perturbative relations among correlation functions. We discuss Hopf algebra symmetries and such relations in braided quantum field theories. We give the four algebraic conditions between Hopf algebra symmetries and braided quantum field theories, which are required for the relations to hold. As concrete examples, we apply our discussions to the Poincaré symmetries of two examples of noncommutative field theories. One is the effective quantum field theory of three-dimensional quantum gravity coupled with spinless particles given by Freidel and Livine, and the other is noncommutative field theory on Moyal plane. We also comment on quantum field theory on  $\kappa$ -Minkowski spacetime.

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<sup>\*</sup> e-mail: sasai@yukawa.kyoto-u.ac.jp

<sup>†</sup> e-mail: sasakura@yukawa.kyoto-u.ac.jp

# 1 Introduction

Symmetry is one of the most important notions in quantum field theory. In many examples, it is useful in investigating properties of quantum field theories non-perturbatively, is a guiding principle in constructing field theories for various purposes such as grand unification, or gives powerful methods in finding exact solutions. It also plays important roles in actual renormalization procedures. Therefore it should be interesting to study symmetries also in noncommutative field theories [1, 2, 3, 4, 5], which may result from some quantum gravity effects [6].

A difficulty in the study in this direction is the apparent violation of basic symmetries such as Poincaré symmetry in the noncommutativity of spacetime. For example, the Moyal plane  $[x^\mu, x^\nu] = i\theta^{\mu\nu}$  is translational invariant, but is not Lorentz or rotational invariant. Another example is the three-dimensional spacetime with noncommutativity  $[x^i, x^j] = i\kappa\epsilon^{ijk}x_k$  ( $i, j, k = 1, 2, 3$ ) [7, 8, 9, 10] with a noncommutativity parameter  $\kappa$ . This noncommutative spacetime is Lorentz-invariant, but is not invariant under the translational transformation  $x^i \rightarrow x^i + a^i$  with  $c$ -number  $a^i$ . In fact, a naive construction of noncommutative quantum field theory on this spacetime leads to rather disastrous violations of energy-momentum conservation [10]: the violations coming from the non-planar diagrams do not vanish in the commutative limit  $\kappa \rightarrow 0$  as in the UV/IR mixing phenomena [11].

In recent years, however, there has been interesting conceptual progress in understanding symmetries in noncommutative field theories: the symmetry transformations in noncommutative spacetime are not the usual Lie-algebraic type, but should be generalized to have Hopf algebraic structures. The Moyal plane was pointed out to be invariant under the twisted Poincaré transformation in [12, 13, 14] and under the twisted diffeomorphism in [15, 16, 17, 18]. There have been various proposals to implement the twisted Poincaré invariance in quantum field theories [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. As for the noncommutative spacetime with  $[x^i, x^j] = i\kappa\epsilon^{ijk}x_k$ , a noncommutative quantum field theory was derived as the effective field theory of three-dimensional quantum gravity with matters [31]. Its essential difference from the naive construction mentioned above is the nontrivial braiding for each crossing in non-planar Feynman diagrams. With this braiding, there exists a kind of conserved energy-momentum in the amplitudes, and the energy-momentum operators have Hopf algebraic structures.

Our aim of this paper is to systematically understand these Hopf algebraic symmetries and their consequences in noncommutative field theories in the framework of braided quantum field theories proposed by Oeckl [34]. In the usual quantum field theories, symmetries give non-perturbative relations among correlation functions. We will see that such relations have natural extensions to the Hopf algebraic symmetries in braided quantum field theories, and will obtain the four conditions for the relations to hold. These conditions should be interpreted as the criteria of the symmetries in braided quantum field theories.

This paper is organized as follows. In the following section, we review braided quantum field theory. This review part follows faithfully the original paper [34], but figures are more

extensively used in the proofs and the explanations to make this paper self-contained and intuitively understandable. We start with braided category and braided Hopf algebra. Then correlation functions of braided quantum field theory are represented in terms of them. Finally braided Feynman rules are given.

In Section 3, we first review the axioms of *action*<sup>1</sup> of an algebra on vector spaces. Then we consider the relations among correlation functions in braided quantum field theory. We find that four algebraic conditions are required for the relations to hold. Then, as concrete examples, we discuss whether the noncommutative field theories mentioned above have the Poincaré symmetry by checking the four conditions. In the former case, we find that the twisted Poincaré symmetry is implemented only after the introduction of a non-trivial braiding factor, which agrees with the previous proposal in [21, 35]. In the latter case, we find that the theory has a kind of translational symmetry, which is different from the usual one by multi-field contributions. We also give some examples of such relations among correlation functions and the implications.

The final section is devoted to summary and comments. We comment on quantum field theory on  $\kappa$ -Minkowski spacetime whose noncommutativity of coordinates is  $[x^0, x^j] = \frac{i}{\kappa}x^j$  ( $j = 1, 2, 3$ ) [36].

## 2 Review of braided quantum field theory

### 2.1 Braided categories and braided Hopf algebras

First of all, we review braided categories and braided Hopf algebras [34, 37]. Braided categories are composed of an object  $X$ , which is a vector space, a dual object  $X^*$ , which is a dual vector space, and morphisms

$$\text{ev} : X^* \otimes X \rightarrow \mathbb{k} \quad (\text{evaluation}), \quad (1)$$

$$\text{coev} : \mathbb{k} \rightarrow X \otimes X^* \quad (\text{coevaluation}), \quad (2)$$

where  $\mathbb{k}$  is a c-number. The composition of the two morphisms in an obvious way makes the identity. Then the braided categories have also an invertible morphism

$$\psi_{V,W} : V \otimes W \rightarrow W \otimes V \quad (\text{braiding}), \quad (3)$$

where  $V, W$  are any pair of vector spaces. Generally the inverse of braiding is not equal to the braiding itself.

The braiding is required to be compatible with the tensor product such that

$$\begin{aligned} \psi_{U,V \otimes W} &= (\text{id} \otimes \psi_{U,W}) \circ (\psi_{U,V} \otimes \text{id}), \\ \psi_{U \otimes V,W} &= (\psi_{U,W} \otimes \text{id}) \circ (\text{id} \otimes \psi_{V,W}). \end{aligned} \quad (4)$$

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<sup>1</sup>We use the italic symbol to distinguish it from the action  $S$ .

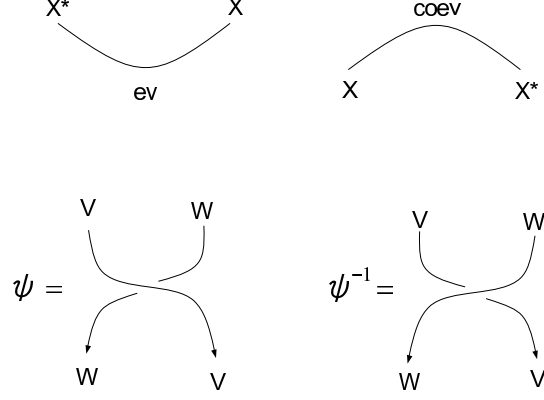


Figure 1: The evaluation, coevaluation, braiding and its inverse.

Then the braiding is also required to be *intersectional* under any morphisms in a Hopf algebra. For example,

$$\begin{aligned}\psi_{Z,W}(Q \otimes \text{id}) &= (\text{id} \otimes Q)\psi_{V,W} \quad \text{for any } Q : V \rightarrow Z, \\ \psi_{V,Z}(\text{id} \otimes Q) &= (Q \otimes \text{id})\psi_{V,W} \quad \text{for any } Q : W \rightarrow Z,\end{aligned}\tag{5}$$

where  $Z$  is a vector space.

We can represent these axioms in pictorial ways [38]. We write the morphisms,  $\text{ev}$ ,  $\text{coev}$ ,  $\psi$ , downwards as in Figure 1. Thus the axioms (4) are represented as in Figure 2, and the axioms (5) are represented as in Figure 3.

Next we consider the polynomials of  $X$ ,

$$\widehat{X} := \bigoplus_{n=0}^{\infty} X^n, \quad \text{with } X^0 := \mathbf{1} \text{ and } X^n := \underbrace{X \otimes \cdots \otimes X}_{n \text{ times}},\tag{6}$$

where  $\mathbf{1}$  is the trivial one-dimensional space.  $\widehat{X}$  naturally has the structure of a braided Hopf algebra via

$$\cdot \text{ (product)} : \widehat{X} \hat{\otimes} \widehat{X} \rightarrow \widehat{X},\tag{7}$$

$$\eta \text{ (unit)} : \mathbb{k} \rightarrow \widehat{X} ; \eta(1) = \mathbf{1},\tag{8}$$

$$\Delta \text{ (coproduct)} : \widehat{X} \rightarrow \widehat{X} \hat{\otimes} \widehat{X} ; \Delta\phi = \phi \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \phi, \text{ and } \Delta(\mathbf{1}) = \mathbf{1} \hat{\otimes} \mathbf{1},\tag{9}$$

$$\epsilon \text{ (counit)} : \widehat{X} \rightarrow \mathbb{k} ; \epsilon(\phi) = 0, \text{ and } \epsilon(\mathbf{1}) = 1,\tag{10}$$

$$S \text{ (antipode)} : \widehat{X} \rightarrow \widehat{X} ; S\phi = -\phi, \text{ and } S(\mathbf{1}) = \mathbf{1},\tag{11}$$

where  $\phi \in X$ . The tensor product  $\otimes$  is the same as the usual product of  $X$ s, while the new tensor product  $\hat{\otimes}$  is the tensor product of  $\widehat{X}$ s. The coproduct  $\Delta$ , counit  $\epsilon$ , antipode  $S$  of the

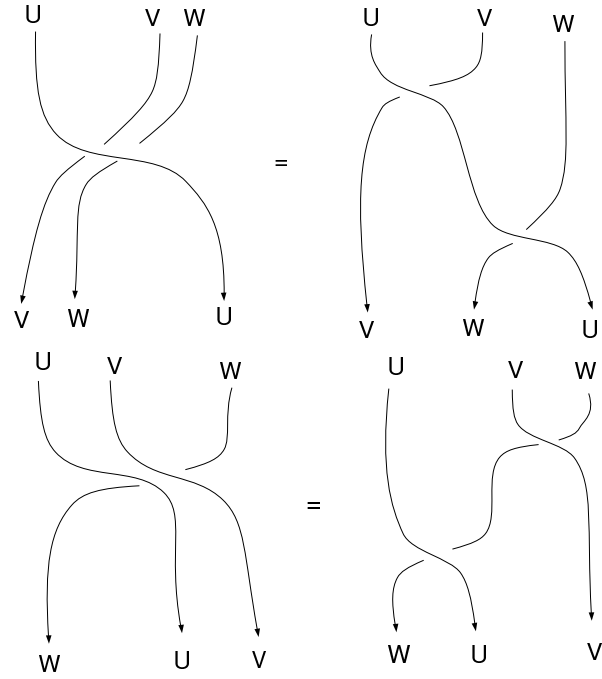


Figure 2: The axioms of braiding (4).

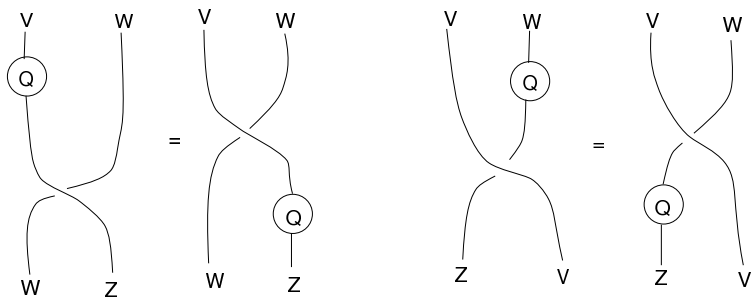


Figure 3: The axioms of braiding (5).

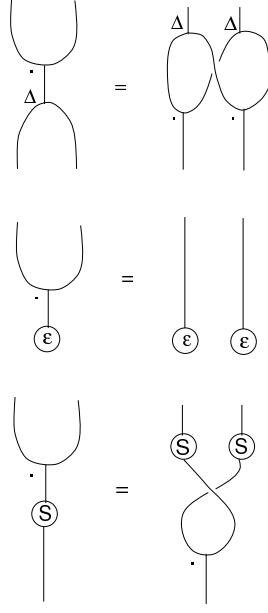


Figure 4: The axioms of coproduct, counit, antipode for products.

products of  $X$ s are defined inductively by

$$\Delta \circ \cdot = (\cdot \hat{\otimes} \cdot) \circ (\text{id} \hat{\otimes} \psi \hat{\otimes} \text{id}) \circ (\Delta \hat{\otimes} \Delta), \quad (12)$$

$$\epsilon \circ \cdot = \cdot \circ (\epsilon \hat{\otimes} \epsilon), \quad (13)$$

$$S \circ \cdot = \cdot \circ \psi \circ (S \hat{\otimes} S). \quad (14)$$

These axioms are diagrammatically represented in Figure 4.

## 2.2 Braided quantum field theory

Next we represent braided quantum field theory [34] in terms of the braided category and the braided Hopf algebra. We take the vector space  $X$  as the space of a field  $\phi(x)$ , where  $x$  denotes a general index for independent modes of the field. Thus  $\hat{X}$  is the space of polynomials of the fields such as  $\phi(x_1)\phi(x_2)\cdots\phi(x_n)$ , and  $\mathbf{1}$  correspond to the constant field of unit. We also take the dual vector space  $X^*$  as the space of differentials  $\delta/\delta\phi(x)$ . We take the evaluation and the coevaluation as follows,

$$\text{ev} : \frac{\delta}{\delta\phi(x)} \otimes \phi(x') \rightarrow \delta(x - x'), \quad (15)$$

$$\text{coev} : \mathbf{1} \rightarrow \int_x \phi(x) \otimes \frac{\delta}{\delta\phi(x)}, \quad (16)$$

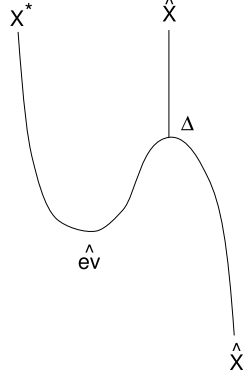


Figure 5: The differentials on  $\hat{X}$ .

where the distribution and the integration should symbolically be understood, and their detailed forms, which may contain non-trivial measures, depend on each case.

The differential on  $\hat{X}$  is defined by

$$\text{diff} := (\hat{\text{ev}} \otimes \text{id}) \circ (\text{id} \otimes \Delta); \quad X^* \otimes \hat{X} \rightarrow \hat{X}, \quad (17)$$

where

$$\hat{\text{ev}}|_{X^* \otimes X^n} = \begin{cases} \text{ev} & \text{for } n = 1, \\ 0 & \text{for } n \neq 1. \end{cases} \quad (18)$$

Diagrammatically this is given by Figure 5.

To see whether the map  $\text{diff}$  gives really the differential of products, let us compute the differential of  $\phi(x)\phi(y)$  as a simple example using the definition (17). This becomes

$$\begin{aligned} \text{diff} \left( \frac{\delta}{\delta \phi(x')} \otimes \phi(x)\phi(y) \right) &= (\hat{\text{ev}} \otimes \text{id}) \circ (\text{id} \otimes \Delta) \left( \frac{\delta}{\delta \phi(x')} \otimes \phi(x)\phi(y) \right) \\ &= (\hat{\text{ev}} \otimes \text{id}) \circ \left( \frac{\delta}{\delta \phi(x')} \otimes \Delta(\phi(x)\phi(y)) \right) \\ &= (\hat{\text{ev}} \otimes \text{id}) \circ \left( \frac{\delta}{\delta \phi(x')} \otimes (\phi(x)\phi(y) \hat{\otimes} \mathbf{1} \right. \\ &\quad \left. + \phi(x) \hat{\otimes} \phi(y) + \psi(\phi(x) \hat{\otimes} \phi(y)) + \mathbf{1} \hat{\otimes} \phi(x)\phi(y)) \right) \\ &= \delta(x' - x) \otimes \phi(y) + (\hat{\text{ev}} \otimes \text{id}) \circ \left( \frac{\delta}{\delta \phi(x')} \otimes \psi(\phi(x) \hat{\otimes} \phi(y)) \right), \end{aligned} \quad (19)$$

where we have used the axiom (12) in deriving the third line. If the braiding is trivial, we find that the differential (17) satisfies the usual Leibniz rule.

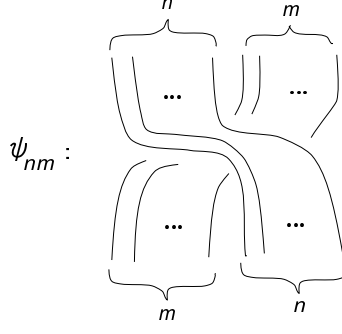


Figure 6: Diagram of  $\psi_{n,m}$ .

Generally we find a braided Leibniz rule

$$\partial(\alpha\beta) = \partial(\alpha)\beta + \psi^{-1}(\partial \otimes \alpha)(\beta) \quad (20)$$

and

$$\partial(\alpha) = (\text{ev} \otimes \text{id}^{n-1})(\partial \otimes [n]_\psi \alpha), \quad (21)$$

where  $\partial \in X^*$ ,  $\alpha, \beta \in \widehat{X}$ , and we have used a simplified notation

$$\partial(\alpha) := \text{diff}(\partial \otimes \alpha). \quad (22)$$

Here  $n$  is the degree of  $\alpha$ , and  $[n]_\psi$  is called a braided integer defined by

$$[n]_\psi := \text{id}^n + \psi \otimes \text{id}^{n-2} + \cdots + \psi_{n-2,1} \otimes \text{id} + \psi_{n-1,1}, \quad (23)$$

where  $\psi_{n,m}$  is a braiding morphism given in Figure 6.

The proofs of the formula (20), (21) are in Appendix A.

Now we define a Gaussian integration, which defines the path integral. The definition is given by

$$\int \partial(\alpha w) := 0 \quad \text{for } \partial \in X^*, \alpha \in \widehat{X}, \quad (24)$$

where  $w \in \widehat{X}$  is a Gaussian weight. In field theory,  $w$  is the exponential of the free part of the action,  $e^{-S_0}$ .

In order to obtain a formula for correlation functions, we define a morphism  $\gamma : X^* \rightarrow X$  such that

$$\partial(w) := -\gamma(\partial)w. \quad (25)$$

This morphism is assumed to be commutative with the braiding as in (5). If  $w = e^{-S_0}$ ,  $\gamma(\partial) = \partial(S_0)$ . In field theory, this is the kinetic part of the action, or the inverse of the propagator.



Starting from (24), we can represent correlation functions of a free field theory in terms of the braided category and the braided Hopf algebra. This is the analog of the Wick theorem in braided quantum field theory. The definition of the free  $n$  point correlation function is given by

$$Z_n^{(0)}(\alpha) := \frac{\int \alpha w}{\int w}, \quad (26)$$

where the degree of  $\alpha$  is  $n$ . Algebraically, this is given by

$$Z_2^{(0)} = \text{ev} \circ (\gamma^{-1} \otimes \text{id}) \circ \psi, \quad (27)$$

$$Z_{2n}^{(0)} = (Z_2^{(0)})^n \circ [2n - 1]_\psi', \quad (28)$$

$$Z_{2n-1}^{(0)} = 0, \quad (29)$$

where

$$[2n - 1]_\psi' := ([1]_\psi' \otimes \text{id}^{2n-1}) \circ ([3]_\psi' \otimes \text{id}^{2n-3}) \circ \cdots \circ ([2n - 1]_\psi' \otimes \text{id}), \quad (30)$$

$$\begin{aligned} [n]_\psi' &:= \text{id}^n + \text{id}^{n-2} \otimes \psi^{-1} + \cdots + \psi_{1,n-1}^{-1} \\ &= \psi_{1,n-1}^{-1} \circ [n]_\psi. \end{aligned} \quad (31)$$

The proofs of (27), (28), (29) are in Appendix B.

Next we consider correlation functions with the existence of an interaction. For  $S = S_0 + \lambda S_{int}$ , a correlation function is perturbatively given by

$$\begin{aligned} Z_n(\alpha) &= \frac{\int \alpha e^{-S}}{\int e^{-S}} \\ &= \frac{\int \alpha (1 - \lambda S_{int} + \cdots) e^{-S_0}}{\int (1 - \lambda S_{int} + \cdots) e^{-S_0}}, \end{aligned} \quad (32)$$

where  $\alpha \in X^n$ . Introducing a morphism  $S_{int} : \mathbb{k} \rightarrow X^k$ , where  $k$  is the degree of  $S_{int}$ , the correlation function is algebraically given by

$$Z_n = \frac{Z_n^{(0)} - \lambda Z_{n+k}^{(0)} \circ (\text{id}^n \otimes S_{int}) + \frac{1}{2} \lambda^2 Z_{n+2k}^{(0)} \circ (\text{id}^n \otimes S_{int} \otimes S_{int}) + \cdots}{1 - \lambda Z_k^{(0)} \circ S_{int} + \frac{1}{2} \lambda^2 Z_{2k}^{(0)} \circ (S_{int} \otimes S_{int}) + \cdots}. \quad (33)$$

Acting  $Z_n$  on  $\alpha \in X^n$ , we obtain the correlation function (32). One can obviously extend  $S_{int}$  to include various interaction terms.

## 2.3 Braided Feynman rules

From the results in the preceding subsection, a correlation function can be represented by summation of diagrams obeying the following rules below.



Figure 7: Propagator (left) and vertex (right).



Figure 8: The braiding  $\psi$  (left) and its inverse  $\psi^{-1}$  (right).

- An  $n$ -point function  $Z_n$  is a morphism  $X^n \rightarrow \mathbb{k}$ . Thus a Feynman diagram starts with  $n$  strands at the top and must be closed at the bottom.
- The propagator  $Z_2^{(0)} : X \otimes X \rightarrow \mathbb{k}$  is represented by the left of Figure 7, which is the abbreviation of Figure 9.
- The interaction vertex  $S_{int} : \mathbb{k} \rightarrow X^k$  is represented by the right of Figure 7. Generally the order of the strands is noncommutative.
- The two kinds of crossings, which are represented in Figure 8, correspond to the braiding and its inverse.
- Any Feynman diagram is built out of propagators, vertices, and crossings, and is closed at the bottom.

### 3 Symmetries in braided quantum field theory

In this section, we discuss symmetries in braided quantum field theory. In order to represent symmetry transformations on fields, we review general description of an *action* in Section



Figure 9: The propagator, which is abbreviated in the left figure of Figure 7.

3.1. In Section 3.2, we study relations among correlation functions. We find four conditions for such relations to follow from a symmetry algebra. In Section 3.3 and 3.4, we treat two examples of (braided) noncommutative field theories and discuss their Poincaré symmetries.

### 3.1 General description of an *action*

We review an *action* of a general Hopf algebra on vector spaces in a mathematical language [37, 39].

An *action*  $\alpha_V$  is a map  $\alpha_V : \mathcal{A} \otimes V \rightarrow V$ , where  $\mathcal{A}$  is an arbitrary Hopf algebra and  $V$  is a vector space (in our case,  $\mathcal{A}$  is a symmetry algebra, and  $V = X$  or  $X^*$ ). We will denote the coproduct and the counit of the Hopf algebra<sup>2</sup> by  $\Delta'$  and  $\epsilon'$  to distinguish them from those of the braided Hopf algebra of fields in Section 2. We do not write all the axioms of an *action*, but our important axioms are the following.

- $\alpha_V$  satisfies the following condition.

$$\alpha_V \circ (\cdot \otimes \text{id}) = \alpha_V \circ (\text{id} \otimes \alpha_V), \quad (34)$$

where the equality acts on  $\mathcal{A} \otimes \mathcal{A} \otimes V$ . This means that  $\alpha_V((a \cdot b) \otimes V) = \alpha_V(a \otimes (\alpha_V(b \otimes V)))$ , where  $a, b \in \mathcal{A}$ . In short we can write this as

$$(a \cdot b) \triangleright V = a \triangleright (b \triangleright V). \quad (35)$$

- An *action* on  $\mathbf{1}$ , which is in a vector space, is defined by

$$\alpha_V(a \otimes \mathbf{1}) = \epsilon'(a)\mathbf{1}, \quad (36)$$

where  $\epsilon'(a)$  is the counit of an algebra  $a \in \mathcal{A}$ .

- An *action* on a tensor product of vector spaces  $V, W$  is defined by

$$\alpha_{V \otimes W}(a) := ((\alpha_V \otimes \alpha_W) \circ \Delta')(a) = \sum_i \alpha_V(a_{(1)}^i) \otimes \alpha_W(a_{(2)}^i), \quad a \in \mathcal{A}, \quad (37)$$

where  $\Delta'(a) = \sum_i a_{(1)}^i \otimes a_{(2)}^i$  is the coproduct of the Hopf algebra  $\mathcal{A}$ . In the case of a usual Lie-algebraic transformation, its coproduct is given by  $\Delta'(a) = a \otimes \mathbf{1} + \mathbf{1} \otimes a$ , where  $\mathbf{1}$  is in  $\mathcal{A}$ . This gives the usual Leibnitz rule.

- Since a Hopf algebra has the coassociativity that

$$((\Delta' \otimes \text{id}) \circ \Delta')(a) = ((\text{id} \otimes \Delta') \circ \Delta')(a), \quad (38)$$

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<sup>2</sup>We omit the antipode.

the *action* on a tensor product of vector spaces, which is obtained by the multiple operations of  $\Delta'$  on  $a$ , is actually unique. An important consequence is that one can divide the *action* on a tensor product of vector spaces as

$$a \triangleright (V_1 \otimes \cdots \otimes V_{k-1} \otimes V_k \otimes \cdots \otimes V_n) = \sum_i a_{(1)}^i \triangleright (V_1 \otimes \cdots \otimes V_{k-1}) \otimes a_{(2)}^i \triangleright (V_k \otimes \cdots \otimes V_n) \quad (39)$$

for any  $k$ .

### 3.2 Symmetry relations among correlation functions and their algebraic descriptions

The expression of the correlation functions (33) is perturbative in interactions, but is a full order algebraic description. Therefore we can discuss the symmetry of the theory and the implied relations among correlation functions by using this expression. We may even expect that the relations will hold non-perturbatively.

In usual quantum field theory, if a field theory has a certain symmetry, there is a relation among the correlation functions in the form,

$$\sum_{i=1}^n \langle \phi(x_1) \cdots \delta_a \phi(x_i) \cdots \phi(x_n) \rangle = 0, \quad (40)$$

where  $\delta_a \phi(x)$  is a variation of a field under a transformation  $a$ , on the assumption that the path integral measure and the action are invariant under the transformation.

If the coproduct of a symmetry algebra is not the usual Lie-algebraic type and thus the Leibniz rule is deformed, the relation will generally have the form,

$$\begin{aligned} & c_a^{(bi)} \langle \phi(x_1) \cdots \delta_b \phi(x_i) \cdots \phi(x_n) \rangle \\ & + c_a^{(bi)(cj)} \langle \phi(x_1) \cdots \delta_b \phi(x_i) \cdots \delta_c \phi(x_j) \cdots \phi(x_n) \rangle \\ & + c_a^{(bi)(cj)(dk)} \langle \phi(x_1) \cdots \delta_b \phi(x_i) \cdots \delta_c \phi(x_j) \cdots \delta_d \phi(x_k) \cdots \phi(x_n) \rangle \\ & + \cdots = 0, \end{aligned} \quad (41)$$

where  $c_a^{\cdots}$  are some coefficients. Its essential difference from (40) is the multi-field contributions. In our algebraic language, the relation can be written as

$$Z_n(a \triangleright \chi) = \epsilon'(a) Z_n(\chi), \quad \text{for } a \in \mathcal{A}, \chi \in X^n. \quad (42)$$

This is equivalent to Figure 10 in our diagrammatic representation. Then we consider what an algebraic structure is required for (42) to hold for any  $a$  and  $\chi$ , i.e. the theory is invariant under the Hopf algebra transformation  $\mathcal{A}$ .

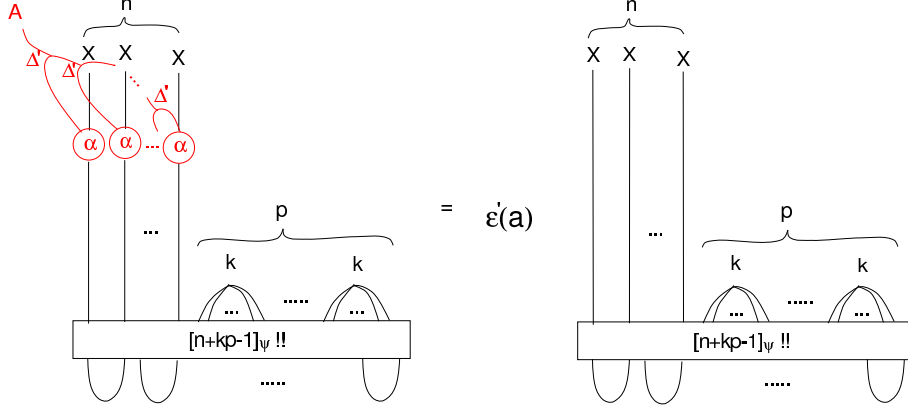


Figure 10: A relation among correlation functions in the diagrammatic representation.  $n$  is the number of external legs,  $k$  is the order of the interaction, and  $p$  is the order of the perturbation.  $n + kp$  is even.

Let us write the coproduct of an element  $a \in \mathcal{A}$  as

$$\Delta'(a) = \sum_s f^s \otimes g^s, \quad (43)$$

where  $f^s, g^s \in \mathcal{A}$ . Since the coproduct must satisfy the Hopf algebra axiom [37],

$$(\epsilon' \otimes \text{id})\Delta'(a) = (\text{id} \otimes \epsilon')\Delta'(a) = a, \quad (44)$$

$f^s, g^s$  must satisfy

$$\sum_s \epsilon'(f^s) \otimes g^s = \sum_s f^s \otimes \epsilon'(g^s) = a. \quad (45)$$

For all the relations among correlation functions to hold, we find the following four conditions for any *action*  $a \in \mathcal{A}$ .

- (Condition 1)  $S_{int}$  must satisfy

$$a \triangleright S_{int} = \epsilon'(a) S_{int}. \quad (46)$$

- (Condition 2) The braiding  $\psi$  is an intertwining operator. That is

$$\psi(a \triangleright (V \otimes W)) = a \triangleright \psi(V \otimes W). \quad (47)$$

- (Condition 3)  $\gamma^{-1}$  and  $a$  are commutative,

$$a \triangleright (\gamma^{-1}(V)) = \gamma^{-1}(a \triangleright V). \quad (48)$$

- (Condition 4) Under an *action*  $a$ , the evaluation map follows

$$\text{ev}(a \triangleright (X^* \otimes X)) = \epsilon'(a) \text{ev}(X^* \otimes X). \quad (49)$$

Condition 1 to 4 are diagrammatically represented in Figure 11. It is clear that, when the algebra  $\mathcal{A}$  is generated from a finite number of its independent elements, it is enough for these generators to satisfy these conditions.

Condition 1 is the requirement of the symmetry at the classical level for the interaction. We can extend this condition to

$$(a \triangleright X^n) \otimes S_{int}^p = a \triangleright (X^n \otimes S_{int}^p). \quad (50)$$

The proof is the following. From a coproduct (43) and its coassociativity (39), the right hand side of (50) is equal to

$$\sum_s (f^s \triangleright (X^n \otimes S_{int}^{p-1})) \otimes g^s \triangleright S_{int} \quad (51)$$

Since Condition 1 implies

$$g^s \triangleright S_{int} = \epsilon'(g^s) S_{int}, \quad (52)$$

(51) becomes

$$\begin{aligned} & \sum_s (f^s \triangleright (X^n \otimes S_{int}^{p-1})) \otimes \epsilon'(g^s) S_{int} \\ &= a \triangleright (X^n \otimes S_{int}^{p-1}) \otimes S_{int}, \end{aligned} \quad (53)$$

where we have used (45). Iterating this procedure, we obtain the left-hand side of (50).

Condition 2,3,4 can also be extended to

$$[n + kp - 1]_\psi!! \circ (a \triangleright X^{n+kp}) = a \triangleright [n + kp - 1]_\psi!! X^{n+kp}, \quad (54)$$

$$(\gamma^{-1} \otimes \text{id})^{\frac{n+pk}{2}} \circ (a \triangleright X^{n+kp}) = a \triangleright (\gamma^{-1} \otimes \text{id})^{\frac{n+pk}{2}} X^{n+kp}, \quad (55)$$

$$\text{ev}^{\frac{n+pk}{2}}(a \triangleright (X^* \otimes X)^{\frac{n+pk}{2}}) = \epsilon'(a) \text{ev}^{\frac{n+pk}{2}}(X^* \otimes X)^{\frac{n+pk}{2}}. \quad (56)$$

We can find that these extended conditions (50), (54), (55), (56) can be represented as in Figure 12. In the diagrammatic language, the relation among correlation functions holds if an *action* can pass downwards through a Feynman diagram and satisfies (36).

### 3.3 Symmetries of the effective noncommutative field theory of three-dimensional quantum gravity coupled with scalar particles

In this subsection, we discuss the Poincaré symmetry of the effective noncommutative field theory of three-dimensional quantum gravity coupled with scalar particles, which was obtained in [31] by studying the Ponzano-Regge model [40] coupled with spinless particles. The

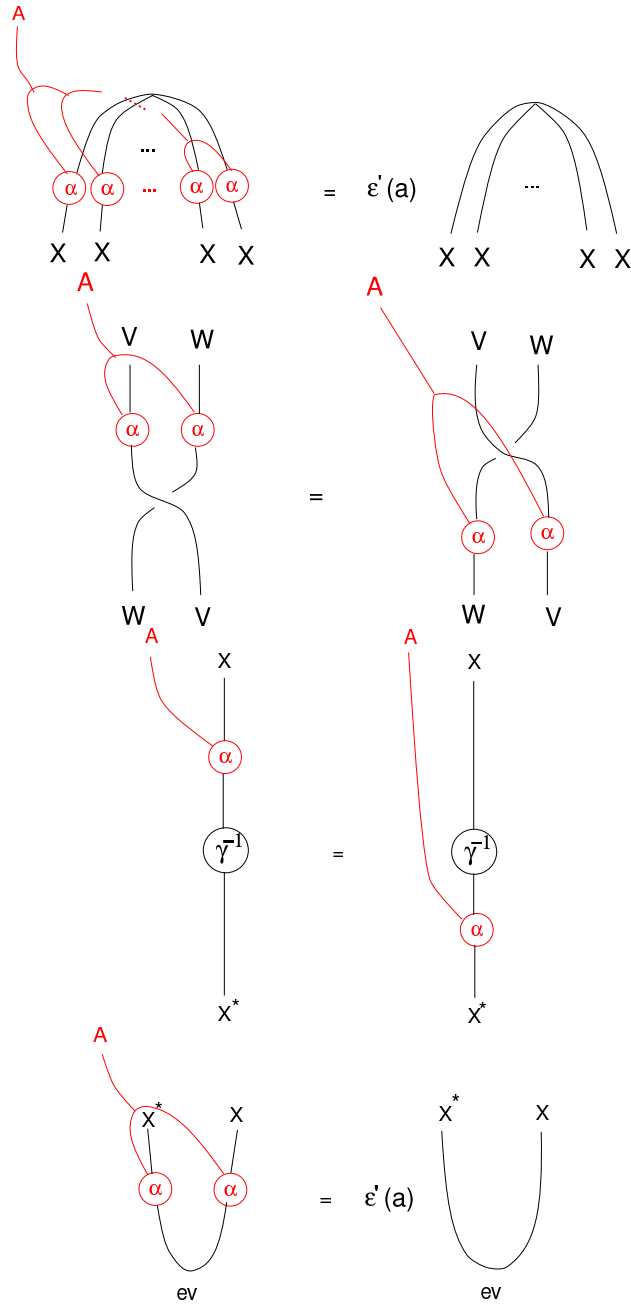


Figure 11: Conditions 1,2,3, and 4.

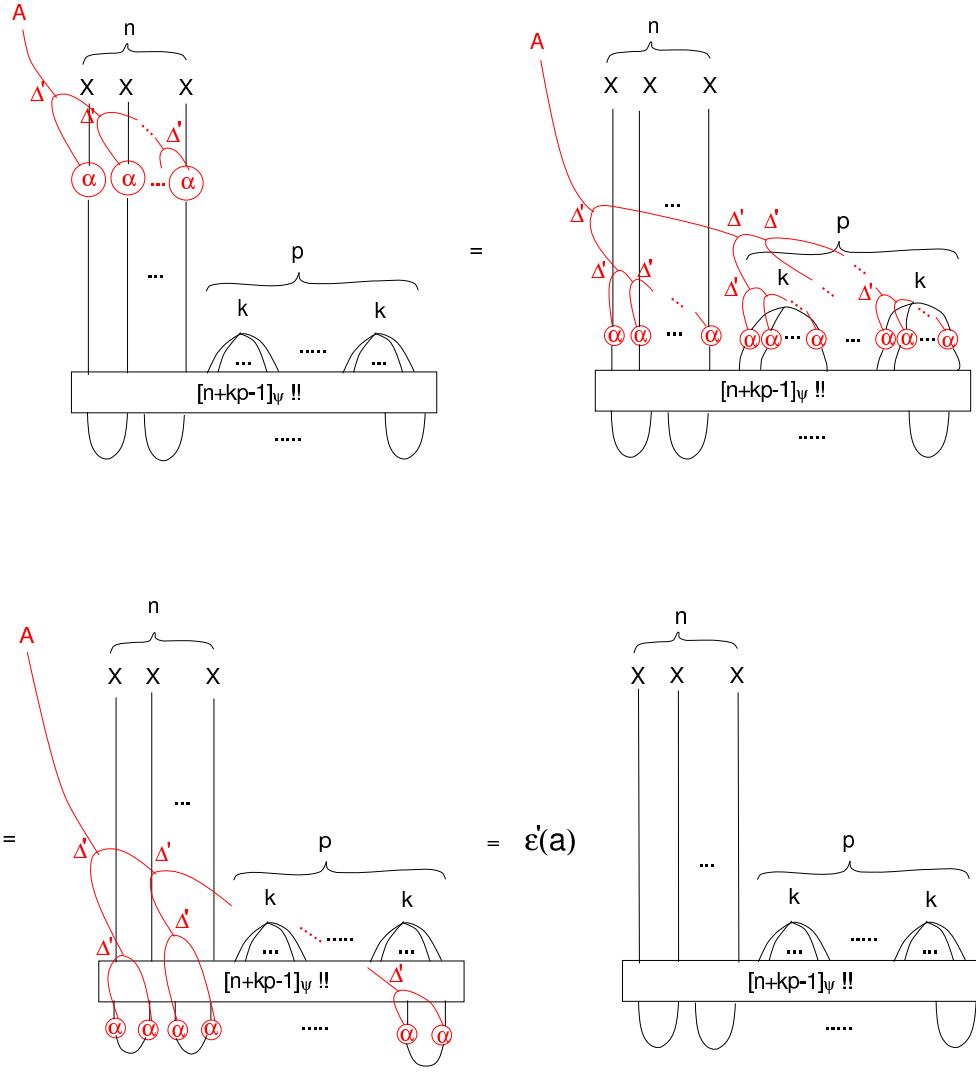


Figure 12: A relation among correlation functions is satisfied if the four conditions (46), (47), (48), (49) are satisfied.



symmetries of this theory is also known as DSU(2), which was discussed in [32, 33]. We first review the field theory [10, 31].

Let  $\phi(x)$  be a scalar field on a three-dimensional space  $x = (x^1, x^2, x^3)$ . Its Fourier transformation is given by

$$\phi(x) = \int dg \tilde{\phi}(g) e^{\frac{i}{2\kappa} \text{tr}(\mathcal{X}g)}, \quad (57)$$

where  $\kappa$  is a constant,  $\mathcal{X} = ix^i \sigma_i$ , and  $g = P^0 - i\kappa P^i \sigma_i \in \text{SO}(3)^3$  with Pauli matrices  $\sigma_i$ . Here  $\int dg$  is the Haar measure of  $\text{SO}(3)$  and  $P^0 = \pm \sqrt{1 - \kappa^2 P^i P_i}$  by definition. In the following discussions, we will only deal with the Euclidean case, but the Lorentzian case can also be treated in a similar manner by replacing  $\text{SO}(3)$  with  $\text{SL}(2, R)$ .

The definition of the star product is given by

$$e^{\frac{i}{2\kappa} \text{tr}(\mathcal{X}g_1)} \star e^{\frac{i}{2\kappa} \text{tr}(\mathcal{X}g_2)} := e^{\frac{i}{2\kappa} \text{tr}(\mathcal{X}g_1 g_2)}. \quad (58)$$

Differentiating both hands sides of (58) with respect to  $P_1^i := P^i(g_1)$  and  $P_2^j := P^j(g_2)$  and then taking the limit  $P_1^i, P_2^j \rightarrow 0$ , one finds the  $\text{SO}(3)$  Lie-algebraic space-time noncommutativity [7, 8, 9],

$$[x^i, x^j]_\star = 2i\kappa \epsilon^{ijk} x_k. \quad (59)$$

For example, the action<sup>4</sup> of a  $\phi^3$  theory is

$$S = \frac{1}{8\pi\kappa^3} \int d^3x \left[ \frac{1}{2} (\partial_i \phi \star \partial_i \phi)(x) - \frac{1}{2} M^2 (\phi \star \phi)(x) + \frac{\lambda}{3!} (\phi \star \phi \star \phi)(x) \right], \quad (60)$$

where  $M^2 = \frac{\sin^2 m\kappa}{\kappa^2}$ . Its momentum representation is

$$S = \frac{1}{2} \int dg (P^2(g) - M^2) \tilde{\phi}(g) \tilde{\phi}(g^{-1}) + \frac{\lambda}{3!} \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) \tilde{\phi}(g_1) \tilde{\phi}(g_2) \tilde{\phi}(g_3), \quad (61)$$

from which it is straightforward to read the Feynman rules.

Some quantum properties of this scalar field theory were analyzed in [10]. As can be seen from (59), the naive translational symmetry is violated. In fact, the violation is rather disastrous. There exists a kind of conserved energy-momentum in the amplitudes of the tree and the planar loop diagrams, but this energy-momentum is not conserved in the non-planar loop diagrams. Moreover, the violation of the energy-momentum conservation does not vanish in the commutative limit  $\kappa \rightarrow 0$  due to a mechanism similar to the UV/IR phenomena [11].

---

<sup>3</sup>The identification  $g \sim -g$  is implicitly assumed.

<sup>4</sup>Since in the Ponzano-Regge model the definition of the weight of partition function is  $e^{iS}$  despite of Euclidean theory, the sign of the mass term is not the usual one.

In the effective field theory of quantum gravity coupled with spinless particles, however, the Feynman rules contain also a non-trivial braiding rule for each crossing, which comes from a flatness condition in a graph of intersecting particles [31]. This can be incorporated as a braiding between the scalar fields,

$$\psi(\tilde{\phi}(g_1)\tilde{\phi}(g_2)) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2), \quad (62)$$

in the braided quantum field theory.

From the direct analysis of the Feynman graphs with this braiding rule, one can easily find that the energy-momentum mentioned above is conserved also in the non-planar diagrams. This suggests the existence of a translational symmetry in the quantum field theory. In the sequel, we will discuss the embedding of this field theory into the framework of braided quantum field theory, and will check the four conditions for its translational and rotational symmetries.

We use the momentum representation, and take  $X$  as the space of  $\tilde{\phi}(g)$  and  $X^*$  as that of  $\frac{\delta}{\delta\tilde{\phi}(g)}$ . We take the braided Hopf algebra of the fields as follows,

$$\Delta : \tilde{\phi}(g) \rightarrow \tilde{\phi}(g) \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \tilde{\phi}(g), \quad (63)$$

$$\epsilon : \tilde{\phi}(g) \rightarrow 0, \quad (64)$$

$$S : \tilde{\phi}(g) \rightarrow -\tilde{\phi}(g), \quad (65)$$

$$\psi : \tilde{\phi}(g_1) \otimes \tilde{\phi}(g_2) \rightarrow \tilde{\phi}(g_2) \otimes \tilde{\phi}(g_2^{-1}g_1g_2). \quad (66)$$

The evaluation and coevaluation maps are given by

$$\text{ev} : \frac{\delta}{\delta\tilde{\phi}(g)} \otimes \tilde{\phi}(g') \rightarrow \delta(g^{-1}g'), \quad (67)$$

$$\text{coev} : \mathbf{1} \rightarrow \int dg \tilde{\phi}(g) \otimes \frac{\delta}{\delta\tilde{\phi}(g)}. \quad (68)$$

From  $\gamma(\partial) = \partial S_0 = (P^2(g) - m^2)\tilde{\phi}(g^{-1})$ ,

$$\gamma^{-1}(\tilde{\phi}(g)) = \frac{1}{P^2(g^{-1}) - m^2} \frac{\delta}{\delta\tilde{\phi}(g^{-1})}. \quad (69)$$

From the algebraic consistencies in Figure 13, the braidings between  $X$  and  $X^*$  and the braiding between  $X^*$ s are determined to be

$$\psi\left(\frac{\delta}{\delta\tilde{\phi}(g_1)} \otimes \tilde{\phi}(g_2)\right) = \tilde{\phi}(g_2) \otimes \frac{\delta}{\delta\tilde{\phi}(g_2^{-1}g_1g_2)}, \quad (70)$$

$$\psi\left(\tilde{\phi}(g_1) \otimes \frac{\delta}{\delta\tilde{\phi}(g_2)}\right) = \frac{\delta}{\delta\tilde{\phi}(g_2)} \otimes \tilde{\phi}(g_2g_1g_2^{-1}), \quad (71)$$

$$\psi\left(\frac{\delta}{\delta\tilde{\phi}(g_1)} \otimes \frac{\delta}{\delta\tilde{\phi}(g_2)}\right) = \frac{\delta}{\delta\tilde{\phi}(g_2)} \otimes \frac{\delta}{\delta\tilde{\phi}(g_2g_1g_2^{-1})}. \quad (72)$$

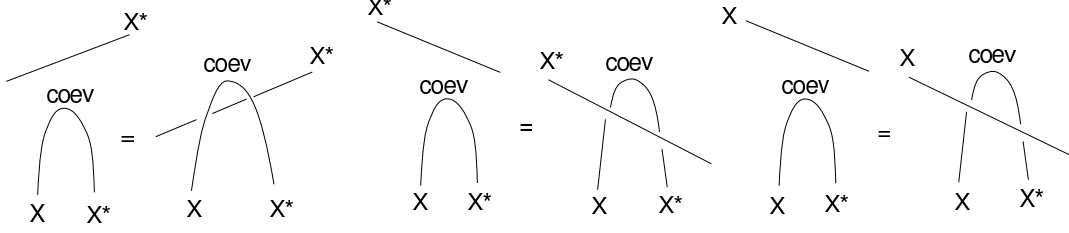


Figure 13: The algebraic consistency conditions of coevaluation map and  $X, X^*$ .

In this derivation, we have used the invariance of the Haar measure  $d(g^{-1}g'g) = dg'$ .

Now we consider a translational transformation of the field. If we shift  $x^i$  to  $x^i + \epsilon^i$ , a field  $\phi(x)$  becomes

$$\begin{aligned}
\phi(x) &\rightarrow \phi(x + \epsilon) \\
&= \int dg \tilde{\phi}(g) e^{i(x+\epsilon)^i P_i(g)} \\
&\sim \int dg (1 + i\epsilon^i P_i(g)) \tilde{\phi}(g) e^{ix^i P_i(g)}.
\end{aligned} \tag{73}$$

Thus in the momentum representation, the translational transformation corresponds to an *action*

$$P^i \triangleright \tilde{\phi}(g) = P^i(g) \tilde{\phi}(g), \quad P^0 \triangleright \tilde{\phi}(g) = P^0(g) \tilde{\phi}(g). \tag{74}$$

From the requirement that the star product (58) conserve a kind of momentum, the *action* on a product of fields should be

$$\begin{aligned}
P^i \triangleright (\tilde{\phi}(g_1) \tilde{\phi}(g_2)) &= P^i(g_1 g_2) \tilde{\phi}(g_1) \tilde{\phi}(g_2) \\
&= (P_1^0 P_2^i + P_2^0 P_1^i + \kappa \epsilon^{ijk} P_1^j P_2^k) \tilde{\phi}(g_1) \tilde{\phi}(g_2),
\end{aligned} \tag{75}$$

$$P^0 \triangleright (\tilde{\phi}(g_1) \tilde{\phi}(g_2)) = (P_1^0 P_2^0 - \kappa^2 P_1^i P_{2i}) \tilde{\phi}(g_1) \tilde{\phi}(g_2). \tag{76}$$

This determines the coproduct of  $P^i, P^0$  as

$$\Delta'(P^i) = P^0 \otimes P^i + P^i \otimes P^0 + \kappa \epsilon^{ijk} P^j \otimes P^k, \tag{77}$$

$$\Delta'(P^0) = P^0 \otimes P^0 - \kappa^2 P^i \otimes P_i. \tag{78}$$

This coproduct satisfies the coassociativity, which essentially comes from the associativity of the group multiplication.

From the axiom (44), the counit of  $P^i, P^0$  is given by

$$\epsilon'(P^i) = \epsilon'(P^0) = 0. \tag{79}$$

Since the conservation of momentum under the coevaluation map (68) requires that the *action* of  $P^i$  on  $\int dg(\tilde{\phi}(g) \otimes \frac{\delta}{\delta\phi(g)})$  vanish from (36), the *action* of  $P^i$  on  $\frac{\delta}{\delta\phi(g)}$  must be

$$P^i \triangleright \frac{\delta}{\delta\tilde{\phi}(g)} = P^i(g^{-1}) \frac{\delta}{\delta\tilde{\phi}(g)}. \quad (80)$$

In the following, we see that the momentum algebra satisfies the four conditions (46), (47), (48), (49).

Condition 1 is satisfied since

$$\begin{aligned} & P^i \triangleright S_{int} \\ &= \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) P^i \triangleright (\tilde{\phi}(g_1) \tilde{\phi}(g_2) \tilde{\phi}(g_3)) \\ &= \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) P^i(g_1 g_2 g_3) (\tilde{\phi}(g_1) \tilde{\phi}(g_2) \tilde{\phi}(g_3)) \\ &= 0. \end{aligned} \quad (81)$$

Condition 2 is satisfied since

$$\begin{aligned} \psi(P^i \triangleright (\tilde{\phi}(g_1) \tilde{\phi}(g_2))) &= P^i(g_1 g_2) (\tilde{\phi}(g_2) \tilde{\phi}(g_2^{-1} g_1 g_2)), \\ P^i \triangleright \psi(\tilde{\phi}(g_1) \tilde{\phi}(g_2)) &= P^i(g_2 g_2^{-1} g_1 g_2) (\tilde{\phi}(g_2) \tilde{\phi}(g_2^{-1} g_1 g_2)). \end{aligned}$$

Condition 3 is satisfied since

$$\begin{aligned} P^i \triangleright \gamma^{-1}(\tilde{\phi}(g)) &= \frac{1}{P^2(g^{-1}) - m^2} P^i(g) \frac{\delta}{\delta\tilde{\phi}(g^{-1})}, \\ \gamma^{-1}(P^i \triangleright \tilde{\phi}(g)) &= \frac{1}{P^2(g^{-1}) - m^2} P^i(g) \frac{\delta}{\delta\tilde{\phi}(g^{-1})}. \end{aligned}$$

Condition 4 is satisfied since

$$\begin{aligned} \text{ev} \left( P^i \triangleright \left( \frac{\delta}{\delta\tilde{\phi}(g_1)} \otimes \tilde{\phi}(g_2) \right) \right) &= P^i(g_1^{-1} g_2) \text{ev} \left( \frac{\delta}{\delta\tilde{\phi}(g_1)} \otimes \tilde{\phi}(g_2) \right) \\ &= 0. \end{aligned} \quad (82)$$

Thus we find that the effective braided noncommutative field theory of three-dimensional quantum gravity coupled with spinless particles has the translational symmetry.

Next we consider a rotational symmetry. The rotational symmetry corresponds to an *action*

$$\Lambda \triangleright \tilde{\phi}(g) = \tilde{\phi}(h^{-1} g h), \quad (83)$$

which is the usual Lie-group one. The *action* on the tensor product is

$$\Lambda \triangleright (\tilde{\phi}(g_1) \otimes \tilde{\phi}(g_2)) = \tilde{\phi}(h^{-1} g_1 h) \otimes \tilde{\phi}(h^{-1} g_2 h). \quad (84)$$

Thus the coproduct of the rotational symmetry is given by

$$\Delta'(\Lambda) = \Lambda \otimes \Lambda. \quad (85)$$

From the axiom (44), the counit of  $\Lambda$  is given by

$$\epsilon'(\Lambda) = 1. \quad (86)$$

Condition 1 is satisfied since

$$\begin{aligned} & \Lambda \triangleright S_{int} \\ &= \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) \Lambda \triangleright (\tilde{\phi}(g_1) \tilde{\phi}(g_2) \tilde{\phi}(g_3)) \\ &= \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) (\tilde{\phi}(h^{-1} g_1 h) \tilde{\phi}(h^{-1} g_2 h) \tilde{\phi}(h^{-1} g_3 h)) \\ &= \epsilon'(\Lambda) S_{int}. \end{aligned} \quad (87)$$

Condition 2 is satisfied since

$$\begin{aligned} \psi(\Lambda \triangleright (\tilde{\phi}(g_1) \otimes \tilde{\phi}(g_2))) &= \tilde{\phi}(h^{-1} g_2 h) \otimes \tilde{\phi}(h^{-1} g_2^{-1} g_1 g_2 h) \\ \Lambda \triangleright \psi(\tilde{\phi}(g_1) \otimes \tilde{\phi}(g_2)) &= \tilde{\phi}(h^{-1} g_2 h) \otimes \tilde{\phi}(h^{-1} g_2^{-1} g_1 g_2 h). \end{aligned} \quad (88)$$

Condition 3 is satisfied since

$$\begin{aligned} \Lambda \triangleright \gamma^{-1}(\tilde{\phi}(g)) &= \frac{1}{P^2(g^{-1}) - m^2} \frac{\delta}{\delta \tilde{\phi}(h^{-1} g^{-1} h)} \\ \gamma^{-1}(\Lambda \triangleright \tilde{\phi}(g)) &= \frac{1}{P^2(h^{-1} g^{-1} h) - m^2} \frac{\delta}{\delta \tilde{\phi}(h^{-1} g^{-1} h)} \\ &= \frac{1}{P^2(g^{-1}) - m^2} \frac{\delta}{\delta \tilde{\phi}(h^{-1} g^{-1} h)}. \end{aligned} \quad (89)$$

Condition 4 is satisfied since

$$\begin{aligned} \text{ev} \left( \Lambda \triangleright \left( \frac{\delta}{\delta \tilde{\phi}(g_1)} \otimes \tilde{\phi}(g_2) \right) \right) &= \text{ev} \left( \frac{\delta}{\delta \tilde{\phi}(h^{-1} g_1 h)} \otimes \tilde{\phi}(h^{-1} g_2 h) \right) \\ &= \delta(g_1^{-1} g_2) \\ &= \epsilon'(\Lambda) \text{ev} \left( \frac{\delta}{\delta \tilde{\phi}(g_1)} \otimes \tilde{\phi}(g_2) \right). \end{aligned} \quad (90)$$

Thus we find that this braided noncommutative field theory has also the rotational symmetry.

### 3.4 Twisted Poincaré symmetry of noncommutative field theory on Moyal plane

In this subsection, we discuss the twisted Poincaré symmetry of noncommutative field theory on Moyal plane  $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ .

For example, the action of a  $\phi^3$  theory is given by

$$S = \int d^D x \left[ \frac{1}{2} (\partial_\mu \phi * \partial^\mu \phi)(x) - \frac{1}{2} m^2 (\phi * \phi)(x) + \frac{\lambda}{3!} (\phi * \phi * \phi)(x) \right], \quad (91)$$

where the star product is given by

$$\phi(x) * \phi(x) = e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu^x \partial_\nu^y} \phi(x) \phi(y) \Big|_{x=y}. \quad (92)$$

In the momentum representation, the action is

$$S = \int d^D p \left[ \frac{1}{2} (p^2 - m^2) \tilde{\phi}(p) \tilde{\phi}(-p) + \frac{\lambda}{3!} \int d^D p_1 d^D p_2 d^D p_3 e^{-\frac{i}{2} p_{1\mu} \theta^{\mu\nu} p_{2\nu}} \delta(p_1 + p_2 + p_3) \tilde{\phi}(p_1) \tilde{\phi}(p_2) \tilde{\phi}(p_3) \right]. \quad (93)$$

We take  $X$  as the space of  $\tilde{\phi}(p)$  and  $X^*$  as that of  $\frac{\delta}{\delta \tilde{\phi}(p)}$ . Then we take the braided Hopf algebra as follows:

$$\Delta : \tilde{\phi}(p) \rightarrow \tilde{\phi}(p) \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \tilde{\phi}(p), \quad (94)$$

$$\epsilon : \tilde{\phi}(p) \rightarrow 0, \quad (95)$$

$$S : \tilde{\phi}(p) \rightarrow -\tilde{\phi}(p). \quad (96)$$

From  $\gamma(\partial) = \partial S_0 = (p^2 - m^2) \tilde{\phi}(-p)$ ,

$$\gamma^{-1}(\tilde{\phi}(p)) = \frac{1}{p^2 - m^2} \frac{\delta}{\delta \tilde{\phi}(-p)}. \quad (97)$$

Let us consider the twisted Poincaré symmetry [12, 13, 14]. The coproduct and the counit of the twisted Poincaré algebra is given by

$$\begin{aligned} \Delta'(P^\mu) &= P^\mu \otimes \mathbf{1} + \mathbf{1} \otimes P^\mu, \\ \epsilon'(P^\mu) &= 0, \\ \Delta'(M^{\mu\nu}) &= M^{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M^{\mu\nu} \\ &\quad - \frac{1}{2} \theta^{\alpha\beta} [(\delta_\alpha^\mu P^\nu - \delta_\alpha^\nu P^\mu) \otimes P_\beta + P_\alpha \otimes (\delta_\beta^\mu P^\nu - \delta_\beta^\nu P^\mu)], \\ \epsilon'(M^{\mu\nu}) &= 0. \end{aligned} \quad (98)$$

Thus the *action* of the twisted Lorentz algebra on the tensor product is

$$\begin{aligned} M^{\mu\nu} \triangleright (\tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2)) &= M^{\mu\nu} \triangleright \tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2) + \tilde{\phi}(p_1) \otimes M^{\mu\nu} \triangleright \tilde{\phi}(p_2) \\ &\quad - \frac{1}{2} \theta^{\alpha\beta} [(\delta_\alpha^\mu P^\nu - \delta_\alpha^\nu P^\mu) \triangleright \tilde{\phi}(p_1) \otimes P_\beta \triangleright \tilde{\phi}(p_2) \\ &\quad + P_\alpha \triangleright \tilde{\phi}(p_1) \otimes (\delta_\beta^\mu P^\nu - \delta_\beta^\nu P^\mu) \triangleright \tilde{\phi}(p_2)], \end{aligned} \quad (99)$$

where  $M^{\mu\nu} \triangleright \tilde{\phi}(p) = i(p^\mu \partial / \partial p_\nu - p^\nu \partial / \partial p_\mu) \tilde{\phi}(p)$  and  $P^\mu \triangleright \tilde{\phi}(p) = p^\mu \tilde{\phi}(p)$ . The *actions* of  $M^{\mu\nu}$  and  $P^\mu$  on  $\frac{\delta}{\delta \tilde{\phi}(p)}$  are

$$M^{\mu\nu} \triangleright \frac{\delta}{\delta \tilde{\phi}(p)} = i(p^\mu \partial / \partial p_\nu - p^\nu \partial / \partial p_\mu) \frac{\delta}{\delta \tilde{\phi}(p)}, \quad (100)$$

$$P^\mu \triangleright \frac{\delta}{\delta \tilde{\phi}(p)} = -p^\mu \frac{\delta}{\delta \tilde{\phi}(p)}. \quad (101)$$

One easily finds that three conditions (46), (48), (49) are satisfied, but (47) is not if the braiding is trivial. In order to keep the invariance, the braiding must be taken as

$$\psi(\tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2)) = e^{i\theta^{\alpha\beta} p_{2\alpha} \otimes p_{1\beta}} (\tilde{\phi}(p_2) \otimes \tilde{\phi}(p_1)). \quad (102)$$

This agrees with the previous proposal [21, 35].

We can easily check that the translational symmetry holds since the coproduct  $\Delta'(P^\mu)$  follows the usual Leibniz rule.

### 3.5 Relations among correlation functions : Examples

Now we have checked, in all orders of perturbation, that the two theories in the preceding sections have symmetry relations among correlation functions implied by the Hopf algebra symmetries. In Section 3.3 we gave how the translational generator acts on a product of fields in (75), (76) in the momentum representation. Since the physical meaning of this Hopf algebra transformation is not so clear, it would be interesting to see explicitly the symmetry relations among correlation functions. The same thing is also true in the case of the twisted Lorentz symmetry in Section 3.4. In this subsection, we work out explicitly some relations among correlation functions in the two theories.

In the effective quantum field theory of quantum gravity, the *action* of the translational generators on a correlation function is given by

$$\langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle \rightarrow i\epsilon^i P_i(g_1 \cdots g_n) \langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle \quad (103)$$

in the momentum representation, where  $\epsilon^i$  is an infinitesimal parameter. Thus we obtain a relation,

$$P_i(g_1 \cdots g_n) \langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle = 0. \quad (104)$$

This is a (modified) momentum conservation law; the correlation function has support only on the vanishing momentum subspace,  $P_i(g_1 \cdots g_n) = 0$ . This all-order relation in the quantum field theory would be a simple but an important implication of the Hopf algebraic translational symmetry. This provides a good example of the physical importance of a Hopf algebraic symmetry: a Hopf algebra symmetry leads to a (modified) conservation law.

It would also be interesting to see the relations in the coordinate representations, where the fields are defined by  $\phi(x) = \int_p e^{ip \cdot x} \tilde{\phi}(p)$ . As explicitly noted in the preceding subsections, we stress that the basis of the spaces  $X$  of the field variables in the path integrals are parameterized in terms of momenta, and that  $\phi(x)$  are defined by some  $c$ -number linear combinations of them. Therefore, an *action*  $a \in \mathcal{A}$  of a symmetry transformation acts as

$$a \triangleright \phi(x) = \int_p e^{ip \cdot x} (a \triangleright \tilde{\phi}(p)), \quad (105)$$

and the symmetry relations of correlation functions can be obtained by some inverse Fourier transformations (with possible non-trivial measures) of those in momentum representations.

For example, in the case of the two point function, after the inverse Fourier transformation, the relation among correlation functions is given by

$$\langle \partial^i \phi(x_1) \phi(x_2) + \phi(x_1) \partial^i \phi(x_2) \rangle = 0, \quad (106)$$

where we have used the relation (104). Interestingly, this is the usual relation in a translationally invariant quantum field theory. In the case of the three point function, however, the relation is given by

$$\begin{aligned} & \langle \partial^i \phi(x_1) \sqrt{1 + \kappa^2 \partial^2} \phi(x_2) \sqrt{1 + \kappa^2 \partial^2} \phi(x_3) + \sqrt{1 + \kappa^2 \partial^2} \phi(x_1) \partial^i \phi(x_2) \sqrt{1 + \kappa^2 \partial^2} \phi(x_3) \\ & + \sqrt{1 + \kappa^2 \partial^2} \phi(x_1) \sqrt{1 + \kappa^2 \partial^2} \phi(x_2) \partial^i \phi(x_3) + i\kappa \epsilon^{ijk} \sqrt{1 + \kappa^2 \partial^2} \phi(x_1) \partial_j \phi(x_2) \partial_k \phi(x_3) \\ & + i\kappa \epsilon^{ijk} \partial_j \phi(x_1) \sqrt{1 + \kappa^2 \partial^2} \phi(x_2) \partial_k \phi(x_3) - i\kappa \epsilon^{ijk} \partial_j \phi(x_1) \partial_k \phi(x_2) \sqrt{1 + \kappa^2 \partial^2} \phi(x_3) \\ & + \kappa^2 \partial_j \phi(x_1) \partial^j \phi(x_2) \partial^i \phi(x_3) - \kappa^2 \partial^i \phi(x_1) \partial_k \phi(x_2) \partial^k \phi(x_3) \\ & + \kappa^2 \partial_k \phi(x_1) \partial^i \phi(x_2) \partial^k \phi(x_3) \rangle = 0. \end{aligned} \quad (107)$$

This is quite a non-trivial relation among correlation functions, and would be hard to find, if the Hopf algebra symmetry in the quantum field theory was not noticed. This would be another interesting example implying the physical importance of a Hopf algebra symmetry. In general, the relation has the form,

$$\left( \sum_{l=1}^n \partial_{x_l i} - i \sum_{l < m}^n \kappa \epsilon^{ijk} \partial_{x_l j} \partial_{x_m k} + \mathcal{O}(\kappa^2) \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0. \quad (108)$$

In the  $\kappa \rightarrow 0$  limit, the relation approaches the usual relation. Thus the Hopf algebra symmetry is a kind of translational symmetry modified by adding  $\kappa$  dependent higher derivative multi-field contributions.



We can proceed in a similar manner for the twisted Lorentz symmetry. We have a general form of such a symmetry relation as

$$M_{\mu\nu} \triangleright \langle \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n) \rangle = 0. \quad (109)$$

In the case of the two point function, the relation is given by

$$\langle (x_1^\mu \partial^\nu - x_1^\nu \partial^\mu) \phi(x_1) \phi(x_2) + \phi(x_1) (x_2^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_2) \rangle = 0, \quad (110)$$

where we have used the momentum conservation. This is the same relation as that in a Lorentz invariant quantum field theory. In the case of the three point function, the relation is given by

$$\begin{aligned} & \langle (x_1^\mu \partial^\nu - x_1^\nu \partial^\mu) \phi(x_1) \phi(x_2) \phi(x_3) \\ & + \phi(x_1) (x_2^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_2) \phi(x_3) + \phi(x_1) \phi(x_2) (x_3^\mu \partial^\nu - x_3^\nu \partial^\mu) \phi(x_3) \\ & + \frac{1}{2} i \theta^{\alpha\mu} (\partial_\alpha \phi(x_1) \partial^\nu \phi(x_2) \phi(x_3) + \partial_\alpha \phi(x_1) \phi(x_2) \partial^\nu \phi(x_3) + \phi(x_1) \partial_\alpha \phi(x_2) \partial^\nu \phi(x_3) \\ & - \partial^\nu \phi(x_1) \partial_\alpha \phi(x_2) \phi(x_3) - \partial^\nu \phi(x_1) \phi(x_2) \partial_\alpha \phi(x_3) - \phi(x_1) \partial^\nu \phi(x_2) \partial_\alpha \phi(x_3)) \\ & - \frac{1}{2} i \theta^{\alpha\nu} (\partial_\alpha \phi(x_1) \partial^\mu \phi(x_2) \phi(x_3) + \partial_\alpha \phi(x_1) \phi(x_2) \partial^\mu \phi(x_3) + \phi(x_1) \partial_\alpha \phi(x_2) \partial^\mu \phi(x_3) \\ & - \partial^\mu \phi(x_1) \partial_\alpha \phi(x_2) \phi(x_3) - \partial^\mu \phi(x_1) \phi(x_2) \partial_\alpha \phi(x_3) - \phi(x_1) \partial^\mu \phi(x_2) \partial_\alpha \phi(x_3)) \rangle = 0. \end{aligned} \quad (111)$$

In general, the relation among correlation functions has the form,

$$((x_{1\mu} \partial_{x_{1\nu}} - x_{1\nu} \partial_{x_{1\mu}}) + \cdots + (x_{n\mu} \partial_{x_{n\nu}} - x_{n\nu} \partial_{x_{n\mu}}) + \mathcal{O}(\theta)) \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0 \quad (112)$$

in the coordinate representation. The leading terms corresponds to the usual Lorentz transformation  $x^\mu \rightarrow x^\mu + \epsilon^{\mu\nu} x_\nu$ .

The above symmetry relations on Moyal plane can be represented in similar manners as the usual commutative cases, if we use star products. In the papers [24, 25, 26, 27, 28, 29, 30], they have pointed out that in coordinate representation, correlation functions on Moyal plane should be defined with star products extended to non-coincident points (see also [43]) instead of usual products since the usual commutative commutation relation  $[x_i^\mu, x_j^\nu] = 0$  ( $i, j = 1, \dots, n$ ) is not invariant under the twisted Poincaré transformation. Carrying out Fourier transformation of the symmetry relation (109) in momentum representation to such a noncommutative coordinate representation, we obtain the symmetry relations in star tensor products. Namely (110) becomes

$$\langle ((x_1^\mu \partial^\nu - x_1^\nu \partial^\mu) \phi(x_1)) * \phi(x_2) + \phi(x_1) * ((x_2^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_2)) \rangle = 0, \quad (113)$$

and (111) becomes

$$\begin{aligned}
& \langle ((x_1^\mu \partial^\nu - x_1^\nu \partial^\mu) \phi(x_1)) * \phi(x_2) * \phi(x_3) \\
& + \phi(x_1) * ((x_2^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_2)) * \phi(x_3) + \phi(x_1) * \phi(x_2) * ((x_3^\mu \partial^\nu - x_3^\nu \partial^\mu) \phi(x_3)) \\
& + \frac{1}{2} i \theta^{\alpha\mu} (\partial_\alpha \phi(x_1) * \partial^\nu \phi(x_2) * \phi(x_3) + \partial_\alpha \phi(x_1) * \phi(x_2) * \partial^\nu \phi(x_3) + \phi(x_1) * \partial_\alpha \phi(x_2) * \partial^\nu \phi(x_3) \\
& - \partial^\nu \phi(x_1) * \partial_\alpha \phi(x_2) * \phi(x_3) - \partial^\nu \phi(x_1) * \phi(x_2) \partial_\alpha * \phi(x_3) - \phi(x_1) * \partial^\nu \phi(x_2) * \partial_\alpha \phi(x_3)) \\
& - \frac{1}{2} i \theta^{\alpha\nu} (\partial_\alpha \phi(x_1) * \partial^\mu \phi(x_2) * \phi(x_3) + \partial_\alpha \phi(x_1) * \phi(x_2) \partial^\mu * \phi(x_3) + \phi(x_1) * \partial_\alpha \phi(x_2) * \partial^\mu \phi(x_3) \\
& - \partial^\mu \phi(x_1) * \partial_\alpha \phi(x_2) * \phi(x_3) - \partial^\mu \phi(x_1) * \phi(x_2) * \partial_\alpha \phi(x_3) - \phi(x_1) * \partial^\mu \phi(x_2) * \partial_\alpha \phi(x_3)) \rangle = 0.
\end{aligned} \tag{114}$$

More generally we can derive the symmetry relations of correlation functions for tensor fields  $\phi_{\alpha_1 \dots \alpha_n}(x) \equiv \partial_{\alpha_1} \dots \partial_{\alpha_n} \phi(x)$ . For example in the case of the three point function of the tensor fields, the symmetry relation becomes

$$\begin{aligned}
& \langle ((M^{1\mu\nu})_{\alpha_1 \dots \alpha_l} \delta_1 \dots \delta_l \phi_{\delta_1 \dots \delta_l}(x_1)) * \phi_{\beta_1 \dots \beta_m}(x_2) * \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& + \phi_{\alpha_1 \dots \alpha_l}(x_1) * ((M^{2\mu\nu})_{\beta_1 \dots \beta_m} \delta_1 \dots \delta_m \phi_{\delta_1 \dots \delta_m}(x_2)) * \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& + \phi_{\alpha_1 \dots \alpha_l}(x_1) * \phi_{\beta_1 \dots \beta_m}(x_2) * ((M^{3\mu\nu})_{\gamma_1 \dots \gamma_n} \delta_1 \dots \delta_n \phi_{\delta_1 \dots \delta_n}(x_3)) \\
& - \frac{1}{2} \theta^{\alpha\mu} [\partial_\alpha \phi_{\alpha_1 \dots \alpha_l}(x_1) * \partial^\nu \phi_{\beta_1 \dots \beta_m}(x_2) * \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad + \partial_\alpha \phi_{\alpha_1 \dots \alpha_l}(x_1) * \phi_{\beta_1 \dots \beta_m}(x_2) * \partial^\nu \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad + \phi_{\alpha_1 \dots \alpha_l}(x_1) * \partial_\alpha \phi_{\beta_1 \dots \beta_m}(x_2) * \partial^\nu \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad - \partial^\nu \phi_{\alpha_1 \dots \alpha_l}(x_1) * \partial_\alpha \phi_{\beta_1 \dots \beta_m}(x_2) * \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad - \partial^\nu \phi_{\alpha_1 \dots \alpha_l}(x_1) * \phi_{\beta_1 \dots \beta_m}(x_2) \partial_\alpha * \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad - \phi_{\alpha_1 \dots \alpha_l}(x_1) * \partial^\nu \phi_{\beta_1 \dots \beta_m}(x_2) * \partial_\alpha \phi_{\gamma_1 \dots \gamma_n}(x_3)] \\
& + \frac{1}{2} \theta^{\alpha\nu} [\partial_\alpha \phi_{\alpha_1 \dots \alpha_l}(x_1) * \partial^\mu \phi_{\beta_1 \dots \beta_m}(x_2) * \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad + \partial_\alpha \phi_{\alpha_1 \dots \alpha_l}(x_1) * \phi_{\beta_1 \dots \beta_m}(x_2) \partial^\mu * \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad + \phi_{\alpha_1 \dots \alpha_l}(x_1) * \partial_\alpha \phi_{\beta_1 \dots \beta_m}(x_2) * \partial^\mu \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad - \partial^\mu \phi_{\alpha_1 \dots \alpha_l}(x_1) * \partial_\alpha \phi_{\beta_1 \dots \beta_m}(x_2) * \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad - \partial^\mu \phi_{\alpha_1 \dots \alpha_l}(x_1) * \phi_{\beta_1 \dots \beta_m}(x_2) * \partial_\alpha \phi_{\gamma_1 \dots \gamma_n}(x_3) \\
& \quad - \phi_{\alpha_1 \dots \alpha_l}(x_1) * \partial^\mu \phi_{\beta_1 \dots \beta_m}(x_2) * \partial_\alpha \phi_{\gamma_1 \dots \gamma_n}(x_3)] \rangle = 0,
\end{aligned} \tag{115}$$

where

$$\begin{aligned}
(M^{\mu\nu})_{\alpha_1 \dots \alpha_n} \beta_1 \dots \beta_n &= (L^{\mu\nu})_{\alpha_1 \dots \alpha_n} \beta_1 \dots \beta_n + (S^{\mu\nu})_{\alpha_1 \dots \alpha_n} \beta_1 \dots \beta_n \\
(L^{\mu\nu})_{\alpha_1 \dots \alpha_n} \beta_1 \dots \beta_n &= i(x^\mu \partial^\nu - x^\nu \partial^\mu) \delta_{\alpha_1}^{\beta_1} \dots \delta_{\alpha_n}^{\beta_n} \\
(S^{\mu\nu})_{\alpha_1 \dots \alpha_n} \beta_1 \dots \beta_n &= i(\eta^{\nu\beta_1} \delta_{\alpha_1}^{\mu} \delta_{\alpha_2}^{\beta_2} \dots \delta_{\alpha_n}^{\beta_n} - \eta^{\mu\beta_1} \delta_{\alpha_1}^{\nu} \delta_{\alpha_2}^{\beta_2} \dots \delta_{\alpha_n}^{\beta_n})
\end{aligned} \tag{116}$$

If we bring the operators  $(M^{i\mu\nu})_{\alpha_1 \dots \alpha_n}{}^{\beta_1 \dots \beta_n}$  ( $i = 1, 2, 3$ ) out of the star products,  $\theta^{\mu\nu}$  dependent terms are canceled. The final expressions are just the usual Lorentz rotations on the coordinates and the tensorial indices in the correlation functions. This is fully consistent with the discussions in [29].

### 3.6 Origin of Hopf algebra symmetries

To study more the meaning of these additional terms, let us see closer the transformation properties of the star products. In the latter case, it is known that the  $\theta^{\mu\nu}$  dependence of the twisted Lorentz transformation (99) comes from the Lorentz transformation of  $\theta^{\mu\nu}$  itself [41]. To see this, let us consider an infinitesimal Lorentz transformation,  $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$ . The transformation of  $\theta^{\mu\nu}$  is given by

$$\begin{aligned}\theta^{\mu\nu} &\rightarrow \theta^{\mu\nu} + \epsilon^\mu{}_\rho \theta^{\rho\nu} + \epsilon^\nu{}_\rho \theta^{\mu\rho} \\ &:= \theta^{\mu\nu} + \delta\theta^{\mu\nu}.\end{aligned}\tag{117}$$

If one considers not only the transformation of the coordinates,  $x'^\mu = x^\mu + \epsilon^{\mu\nu} x_\nu$ , but also (117), and assumes that  $\phi(x) *_\theta \phi(x)$  and  $\phi'(x') *_{\theta+\delta\theta} \phi'(x')$  be equal, one obtains, after the Fourier transformation,

$$\begin{aligned}\tilde{\phi}'(p_1) \otimes \tilde{\phi}'(p_2) &= \left(1 - \frac{i}{2}(\epsilon^{\mu\nu} M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon^{\mu\nu} M_{\mu\nu} + \delta\theta^{\mu\nu} P_\mu \otimes P_\nu)\right) \tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2) \\ &= \left(1 - \frac{i}{2}\epsilon^{\mu\nu} \Delta' M_{\mu\nu}\right) \tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2),\end{aligned}\tag{118}$$

which agrees with (99). This shows that the additional part of the coproduct of  $M_{\mu\nu}$  takes into account the transformation of the non-dynamical background parameter  $\theta^{\mu\nu}$ .

The former case can be discussed in a similar manner. The definition of the star product is given by

$$e^{ix^i P_i(g_1)} \star_x e^{ix^i P_i(g_2)} = e^{ix^i P_i(g_1 g_2)},\tag{119}$$

where we have explicitly indicated the coordinate where the star product is taken. Then we recognize that  $e^{i(x+\epsilon)^i P_i(g_1)} \star_{x+\epsilon} e^{i(x+\epsilon)^i P_i(g_2)}$  and  $e^{i(x+\epsilon)^i P_i(g_1)} \star_x e^{i(x+\epsilon)^i P_i(g_2)}$  give distinct values. Namely, if the coordinate of the star product is also shifted,

$$e^{i(x+\epsilon)^i P_i(g_1)} \star_{x+\epsilon} e^{i(x+\epsilon)^i P_i(g_2)} = e^{i(x+\epsilon)^i P_i(g_1 g_2)},\tag{120}$$

but, if not,

$$e^{i(x+\epsilon)^i P_i(g_1)} \star_x e^{i(x+\epsilon)^i P_i(g_2)} = e^{i\epsilon^i P_i(g_1)} e^{i\epsilon^i P_i(g_2)} e^{ix^i P_i(g_1 g_2)}.\tag{121}$$

Therefore, if we take the translational transformation as (120), and carry out the same procedure in deriving (59), we always obtain a translational invariant commutation relation<sup>5</sup>,

$$[(x + \epsilon)^i, (x + \epsilon)^j]_{\star_{x+\epsilon}} = 2i\kappa\epsilon^{ijk}(x + \epsilon)_k.\tag{122}$$

---

<sup>5</sup>There is a similar discussion in [42].

Now, assuming that  $\phi(x) \star_x \phi(x)$  and  $\phi'(x') \star_{x'} \phi'(x')$  be equal under the translation  $x^i \rightarrow x'^i = x^i + \epsilon^i$ , we obtain, after the Fourier transformation,

$$\tilde{\phi}'(g_1)\tilde{\phi}'(g_2) = (1 - i\epsilon^i P_i(g_1 g_2))\tilde{\phi}(g_1)\tilde{\phi}(g_2), \quad (123)$$

which is the same as (75).

From these two examples, we anticipate that the multi-field contributions in (41) comes from the transformation properties of the star products.

## 4 Summary and comments

We have discussed symmetries in noncommutative field theories in the framework of braided quantum field theory. We have obtained the algebraic conditions for a Hopf algebra to be a symmetry of a braided quantum field theory, by discussing the conditions for the relations among correlation functions generated from the transformation algebra to hold. Then we have applied our discussions to the Poincaré symmetries in the effective noncommutative field theory of three-dimensional quantum gravity coupled with spinless particles and in the noncommutative field theory on Moyal plane. In the former case we can understand the braiding between fields, which was derived from the three-dimensional quantum gravity computation, from the viewpoint of the translational symmetry of the noncommutative field theory on a Lie-algebraic noncommutative spacetime. In the latter case we have found that the twisted Lorentz symmetry on Moyal plane is a symmetry of the quantum field theory only after the inclusion of the nontrivial braiding factor, which is in agreement with the previous proposal [28, 35]. Then we have discussed the meaning of the Hopf algebra symmetries from the viewpoint of coordinate representation.

In the recent research a noncommutative field theory on  $\kappa$ -Minkowski spacetime is discussed [36]. Since this noncommutativity of the coordinates is given by  $[x^0, x^j] = \frac{i}{\kappa}x^j$ , this noncommutative field theory will not have the naive translational symmetry. We may introduce a non-trivial braiding between fields as in the effective field theory discussed in Section 3.3 to keep the momentum conservation. However, while the effective field theory has the braided category structure because of the invariance of the Haar measure  $d(g^{-1}g'g) = dg'$ , the measure of the momentum space of the field theory on  $\kappa$ -Minkowski spacetime is only left-invariant [36]. Therefore it is not clear to us whether we can embed this field theory on  $\kappa$ -Minkowski spacetime into the framework of braided quantum field theory.

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## A The proofs of the formula (20), (21)

We give the proofs of the formula (20), (21) using diagrams. At first we use the formula

$$\widehat{\text{ev}}(\partial \otimes \alpha\beta) = \widehat{\text{ev}}(\partial \otimes \alpha)\epsilon(\beta) + \widehat{\text{ev}}(\partial \otimes \beta)\epsilon(\alpha), \quad (124)$$

where  $\alpha, \beta \in \widehat{X}$ . This is clear from the definition of  $\widehat{\text{ev}}$ .

Figure 14 gives the proof of (20). In the first line, we use the axiom (12), and in the second line we use the lemma (124). We find the last line from the property of counit.

Next we prove (21). By using the braided Leibniz rule (20) as  $\alpha \in X \otimes \widehat{X}$ , the left-hand side of (21) becomes Figure 15. The first term of Figure 15 becomes  $(\text{ev} \otimes \text{id}^{n-1})(\partial \otimes \text{id}^n \alpha)$  by using the definition of coproduct (9).

In the second term of Figure 15, we divide  $\widehat{X}$  into  $X \otimes \widehat{X}$  and iterate the same as we did above. For example, if the degree of  $\widehat{X}$  is 3, the second term of Figure 15 can be reduced as in Figure 16. We have used  $\Delta X = X \hat{\otimes} 1 + 1 \hat{\otimes} X$  in the second line of Figure 16. The result agrees with (21).

In the same way, we can obtain the formula (21) in general.

## B The proofs of (27), (28), (29)

From the definition of  $\gamma$  (25), we find that

$$\alpha a w = -\alpha \text{diff}(\gamma^{-1}(a) \otimes w), \quad (125)$$

for  $a \in X$  and  $\alpha \in \widehat{X}$ . On the other hand, adding  $\gamma^{-1}$  and  $\psi$  to the braided Leibniz rule (20) as in Figure 17, we find that

$$\alpha \text{diff}(\gamma^{-1}(a) \otimes w) = \text{diff}(\psi(\alpha \otimes \gamma^{-1}(a))w) - (\text{diff} \circ \psi(\alpha \otimes \gamma^{-1}(a)))w. \quad (126)$$

Combining (125), (126), we obtain

$$\alpha a w = -\text{diff}(\psi(\alpha \otimes \gamma^{-1}(a))w) + (\text{diff} \circ \psi(\alpha \otimes \gamma^{-1}(a)))w. \quad (127)$$

Integrating the both hand sides of (127) and using (24), we find that

$$Z^{(0)}(\alpha a) = Z^{(0)}(\text{diff} \circ \psi(\alpha \otimes \gamma^{-1}(a))). \quad (128)$$

If  $\alpha$  is  $b \in X$ ,

$$\begin{aligned} Z^{(0)}(ba) &= Z^{(0)}(\text{diff} \circ \psi(b \otimes \gamma^{-1}(a))) \\ &= \text{ev} \circ \psi(b \otimes \gamma^{-1}(a)) \\ &= \text{ev} \circ (\gamma^{-1} \otimes \text{id}) \circ \psi(b \otimes a). \end{aligned} \quad (129)$$

The figure shows a sequence of five string diagrams connected by equals and plus signs, illustrating the proof of equation (20). The diagrams are as follows:

- Diagram 1:** A string labeled  $X^*$  enters from the top left. It meets a junction where a string labeled  $\hat{X}$  enters from the top. This junction is labeled  $\Delta$ . From this junction, a string labeled  $\hat{X}$  exits to the top right, and another string labeled  $\hat{X}$  exits downwards. The downward string is labeled  $\hat{ev}$ .
- Diagram 2:** Similar to Diagram 1, but the downward string labeled  $\hat{X}$  forms a loop. The loop is labeled  $\Delta$ . The downward string is labeled  $\hat{X}$ .
- Diagram 3:** Similar to Diagram 2, but the loop is labeled  $\epsilon$ . The downward string is labeled  $\hat{X}$ .
- Diagram 4:** Similar to Diagram 3, but the loop is labeled  $\epsilon$ . The downward string is labeled  $\hat{X}$ .
- Diagram 5:** Similar to Diagram 4, but the loop is labeled  $\hat{ev}$ . The downward string is labeled  $\hat{X}$ .

The sequence of transformations is indicated by the symbols:  $=$ ,  $+$ ,  $=$ ,  $+$ .

Figure 14: The proof of (20).

$$\text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3}$$

Figure 15: The left-hand side of (21).

$$\text{Diagram 4} = \text{Diagram 5} + \text{Diagram 6}$$

$$= \text{Diagram 7} + \text{Diagram 8}$$

Figure 16: The second term of Figure 15.



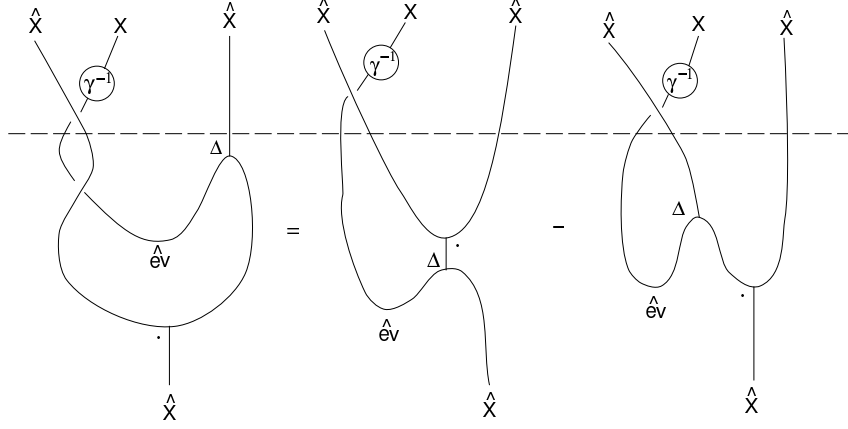


Figure 17: The diagram obtained from adding  $\gamma^{-1}$  and  $\psi$  over the braided Leibniz rule.

Thus we obtain (27).

By putting  $\alpha = \mathbf{1}$ , it is clear that

$$Z_1^{(0)}(a) = 0. \quad (130)$$

Next we rewrite (128) for  $\alpha \in X^{n-1}$  using the formula (21). Diagrammatically it is written as in Figure 18. The second equality is due to (21). Thus we obtain that

$$Z_n^{(0)} = (Z_{n-2}^{(0)} \otimes Z_2^{(0)}) \circ ([n-1]_{\psi}' \otimes \text{id}) \quad (131)$$

Iterating this, we find (28) for even  $n$  and (29) for odd  $n$ .

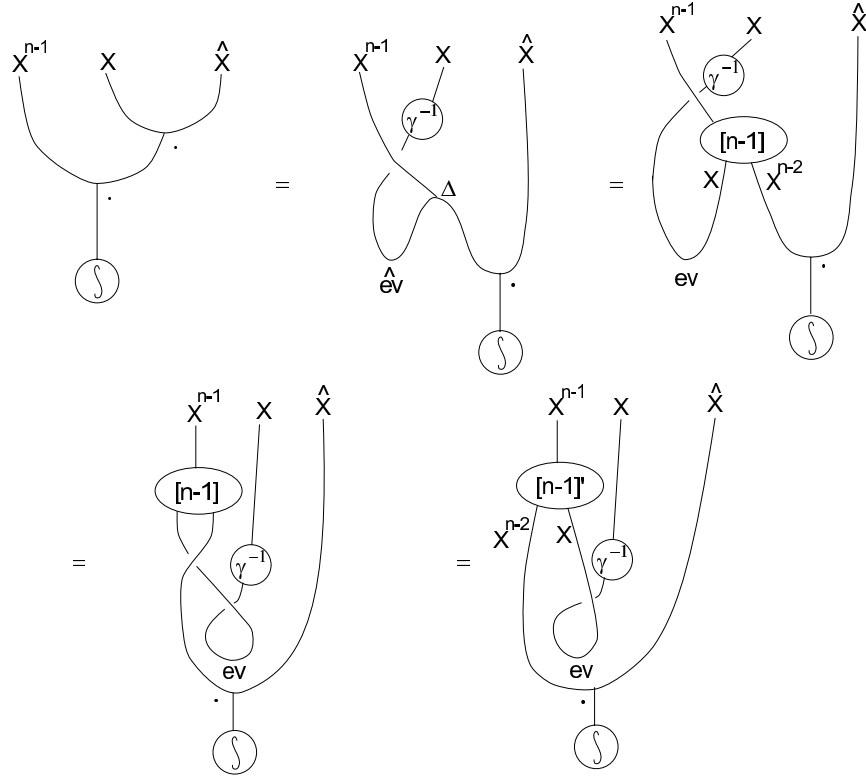


Figure 18: Diagrammatic proof of (131)

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